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Integral representation and tangential limits for monotone BLD functions

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1 Introduction

Our first aim in this paper is to establish an integral representation for BLD functions u in the half space $\mathbf{R}_+^n = \{x = (x_1, ..., x_{n-1}, x_n) : x_n > 0\}$, $n \geqq 2$, such that

$$\int_{\mathbf{R}_+^n} |\nabla^m u(x)|^p dx < \infty,$$

where ∇^m denotes the gradient ∇ iterated m times. Our representation is an extension of Sobolev's integral representation for infinitely differentiable functions with compact support. We give a fine limit result for BLD functions on $\mathbf{R}_+^n$ and then apply the result to the study of tangential limits for monotone BLD functions on $\mathbf{R}_+^n$.

The notion of monotone functions is an extension of monotone functions on the one dimensional space $\mathbf{R}^1$. Harmonic functions together with solutions in a wider class of nonlinear elliptic equations are monotone in our sense; of course, the coordinate functions of quasiregular mappings are monotone.

For $\gamma \geqq 1$, $\xi \in \partial\mathbf{R}_+^n$ and $a > 0$, consider the set

$$T_\gamma(\xi; a) = \{x = (x_1, ..., x_n) \in \mathbf{R}_+^n : |x - \xi|^\gamma < ax_n\}.$$

If $\lim_{x \to \xi, x \in T_\gamma(\xi,a)} u(x) = \ell$ for every $a > 0$, then u is said to have a T_γ-limit ℓ at ξ; u is said to have a nontangential limit at ξ if it has a T_1-limit at ξ. We say further that u has a T_∞-limit ℓ at $\xi \in \partial\mathbf{R}_+^n$ if

$$\lim_{x \to \xi, x \in T_\gamma(\xi,a)} u(x) = \ell$$

for every $\gamma > 1$ and $a > 0$ (cf. [14]).

If u is a monotone function on $\mathbf{R}_+^n$ with finite Dirichlet integral, then we shall show that u has a finite T_∞-limit at every boundary point except for a set $E \subset \partial\mathbf{R}_+^n$ with $C_{1,n}(E) = 0$; see Section 3 for the definition of capacity.

The nontangential case for harmonic functions has been dealt by many mathematicians (cf. Beurling [1], Carleson [2], Gavrilov [4], Wallin [24] and the author [11]). Miklyukov [10] discussed the nontangential limits for quasiregular mappings with finite Dirichlet integral. Recently, Manfredi and Villamor [7] have proved the existence of nontangential limits for monotone functions on the unit ball. The present tangential limit result for harmonic functions was obtained by Cruzeiro [3].

It is well-known (through an application of change of variables) that the coordinate functions of bounded quasiconformal mappings defined on $\mathbf{R}^n_+$ have finite Dirichlet integral. Our theorem then assures the existence of tangential limits for bounded quasiconformal mappings, and thus it gives an affirmative answer to the open problem given by Vuorinen [23, 15.16].

2 Integral representation

For a multi-index $\mu = (\mu_1, ..., \mu_n)$ and a point $x = (x_1, ..., x_n)$, define

$$|\mu| = \mu_1 + \cdots + \mu_n,$$

$$\mu! = \mu_1! \times \cdots \times \mu_n!,$$

$$x^\mu = x_1^{\mu_1} \times \cdots \times x_n^{\mu_n}$$

and

$$D^\mu = \left(\frac{\partial}{\partial x}\right)^\mu = \left(\frac{\partial}{\partial x_1}\right)^{\mu_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\mu_n}.$$

If φ is an infinitely differentiable function on $\mathbf{R}^n$ with compact support, then it is represented as

$$\varphi(x) = \sum_{|\mu|=m} a_\mu \int_{\mathbf{R}^n} \frac{(x-y)^\mu}{|x-y|^n} D^\mu \varphi(y) dy$$

with constants a_μ. This is known as Sobolev's integral representation and is an extension of the representation

$$f(t) = -\frac{1}{(m-1)!} \int_t^\infty (t-s)^{m-1} f^{(m)}(s)\, ds$$

in the one-dimensional case.

To represent general BLD functions in the integral form, we use the kernel functions

$$k_\mu(x) = \frac{x^\mu}{|x|^n}$$

and

$$k_{\mu,\ell}(x,y) = \begin{cases} k_\mu(x-y), & y \in B(0,1), \\ k_\mu(x-y) - \sum_{|\nu| \leqq \ell} \frac{x^\nu}{\nu!}[D^\nu k_\mu](-y), & y \in \mathbf{R}^n - B(0,1). \end{cases}$$

We need the following estimates of the kernel functions.

Lemma 2.1. *If $|x-y| < |x|/2$, then*

$$|k_{\mu,\ell}(x,y)| \leqq M[|x|^{m-n} + |x-y|^{m-n}].$$

Lemma 2.2. *If* $|y| \leqq |x|/2$, *then*

$$|k_{\mu,\ell}(x,y)| \leqq M|x|^{\ell}|y|^{m-n-\ell}.$$

Lemma 2.3. *If* $|y| > 1$ *and* $|y| > 2|x|$, *then*

$$|k_{\mu,\ell}(x,y)| \leqq M|x|^{\ell+1}|y|^{m-n-\ell-1}.$$

Lemma 2.4. *Let* ℓ *be the integer such that*

$$\ell \leqq m - \frac{n}{p} < \ell + 1.$$

For $f \in L^p(\mathbf{R}^n)$ *and* $|\mu| = m$, *set*

$$U_{\mu,\ell}(x) = \int k_{\mu,\ell}(x,y)f(y)dy.$$

Then $\|D^{\nu}U_{\mu,\ell}\|_p \leqq M\|f\|_p$ *for any multi-index* ν *with length* m.

In fact, we first note by Lemma 2.3 and Hölder's inequality that

$$\int_{\{y:|y|>2R\}} |y|^{m-n-\ell-1}|f(y)|dy \leqq \left(\int_{\{y:|y|>2R\}} |y|^{p'(m-n-\ell-1)}dy\right)^{1/p'} \|f\|_p < \infty,$$

where $1/p + 1/p' = 1$. Hence if $|x| < R$, then $U_{\mu,\ell}f$ is of the form

$$U_{\mu,\ell}f(x) = \int_{B(0,2R)} k_{\mu}(x-y)f(y)dy + v(x),$$

where v is an infinitely differentiable function on $B(0,R)$. Further, we see that

$$D^{\nu}(U_{\mu,\ell}f) = (D^{\nu}k_{\mu}) * f + A_{\mu,\nu}f$$

for $|\nu| = m$, where $A_{\mu,\nu}$ is a constant and the convolution on the right-hand side is defined as singular integral. Thus we apply the well-known singular integral theory to obtain the required assertion.

Theorem 2.1 (cf. [17, Theorem 9.2]). *Let* u *be a function in* $L^p_{loc}(\mathbf{R}^n)$ *such that*

$$D^{\mu}u \in L^p(\mathbf{R}^n) \qquad \text{whenever } |\mu| = m;$$

in this case, we write $u \in BL_m(L^p(\mathbf{R}^n))$. *If* ℓ *is the integer such that* $\ell \leqq m - n/p < \ell + 1$, *then there exists a polynomial* P *of degree at most* $m-1$ *such that*

$$u(x) = \sum_{|\mu|=m} \int_{\mathbf{R}^n} k_{\mu,\ell}(x,y)D^{\mu}u(y)dy + P(x) \qquad \text{a.e. on } \mathbf{R}^n.$$

Proof. Denote by U the sum on the right-hand side. In view of Lemmas 2.1, 2.2 and 2.3, we infer that

$$\int |k_{\mu,\ell}(x,y)||D^\mu u(y)|dy$$

is well-defined for almost every x and is locally integrable on $\mathbf{R}^n$. If $\varphi \in C_0^\infty(\mathbf{R}^n)$, then we show that

$$\int \left(\int k_{\mu,\ell}(x,y)D^\mu u(y)dy\right) D^{\nu'+\nu''}\varphi(x)dx = \int \left(\int k_{\mu,\ell}(x,y)D^{\nu'+\nu''}\varphi(x)dx\right) D^\mu u(y)dy$$

whenever $|\nu'| = |\nu''| = m$. For this purpose, consider

$$k_\mu^{(j)}(x) = x^\mu[|x|^2 + (1/j)^2]^{-n/2}$$

and define $k_{\mu,\ell}^{(j)}$ by the same construction as $k_{\mu,\ell}$ from k_μ. Now, if $|\nu'| = |\nu''| = m$ and $\varphi \in C_0^\infty(\mathbf{R}^n)$, then we apply Fubini's theorem to obtain

$$\begin{aligned}
&\int \left(\int k_{\mu,\ell}(x,y)D^\mu u(y)dy\right) D^{\nu'+\nu''}\varphi(x)dx \\
&= \lim_{j\to\infty} \int \left(\int k_{\mu,\ell}^{(j)}(x,y)D^\mu u(y)dy\right) D^{\nu'+\nu''}\varphi(x)dx \\
&= \lim_{j\to\infty} \int \left(\int k_{\mu,\ell}^{(j)}(x,y)D^{\nu'+\nu''}\varphi(x)dx\right) D^\mu u(y)dy \\
&= \lim_{j\to\infty} \int \left((-1)^m \int D^{\nu'}k_\mu^{(j)}(x-y)D^{\nu''}\varphi(x)dx\right) D^\mu u(y)dy \\
&= \lim_{j\to\infty} (-1)^m \int D^{\nu'}k_\mu^{(j)}(z) \left(\int D^{\nu''}\varphi(y+z)D^\mu u(y)dy\right) dz \\
&= \lim_{j\to\infty} (-1)^m \int D^{\nu'}k_\mu^{(j)}(z) \left(\int D^\mu\varphi(y+z)D^{\nu''}u(y)dy\right) dz \\
&= \lim_{j\to\infty} (-1)^m \int \left(\int D^{\nu'}k_\mu^{(j)}(z)D^\mu\varphi(y+z)dz\right) D^{\nu''}u(y)dy \\
&= \lim_{j\to\infty} \int \left(\int k_\mu^{(j)}(x-y)D^{\nu'+\mu}\varphi(x)dx\right) D^{\nu''}u(y)dy \\
&= \int \left(\int k_{\mu,\ell}(x,y)D^{\nu'+\mu}\varphi(x)dx\right) D^{\nu''}u(y)dy.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\int U(x)D^{\nu'+\nu''}\varphi(x)dx &= \int \left(\sum_{|\mu|=m} a_\mu \int k_{\mu,\ell}(x,y)D^{\nu'+\mu}\varphi(x)dx\right) D^{\nu''}u(y)dy \\
&= \int \left(\sum_{|\mu|=m} a_\mu \int k_\mu(x-y)D^{\nu'+\mu}\varphi(x)dx\right) D^{\nu''}u(y)dy \\
&= (-1)^m \int D^{\nu'}\varphi(y)D^{\nu''}u(y)dy \\
&= \int u(y)D^{\nu'+\nu''}\varphi(x)dy.
\end{aligned}$$

Thus we see that $u - U$ is a polynomial of degree at most $2m-1$. Since $D^{\mu}(u-U) \in L^p(\mathbf{R}^n)$, $D^{\mu}(u-U) = 0$, or $u - U$ is a polynomial of degree at most $m-1$.

Next we are concerned with extension properties for BLD functions on the upper half space $\mathbf{R}^n_+$. Let $\lambda_1, \ldots, \lambda_{m+1}$ be a unique solution for the linear system

$$\begin{cases} \lambda_1 + \lambda_2 + \cdots + \lambda_{m+1} = 1, \\ (-1)\lambda_1 + (-2)\lambda_2 + \cdots + (-m-1)\lambda_{m+1} = 1, \\ (-1)^2\lambda_1 + (-2)^2\lambda_2 + \cdots + (-m-1)^2\lambda_{m+1} = 1, \\ \qquad \vdots \\ (-1)^m\lambda_1 + (-2)^m\lambda_2 + \cdots + (-m-1)^m\lambda_{m+1} = 1. \end{cases}$$

For a function $u \in BL_m(L^p(\mathbf{R}^n_+))$, we define

$$Eu(x) = \begin{cases} u(x) & \text{if } x_n > 0, \\ \sum_{j=1}^{m+1} \lambda_j u(x_1, \ldots, x_{n-1}, -jx_n) & \text{if } x_n < 0, \end{cases}$$

and for each multi-index $\mu = (\mu_1, \ldots, \mu_n)$

$$E_\mu u(x) = \begin{cases} u(x) & \text{if } x_n > 0, \\ \sum_{j=1}^{m+1} (-j)^{\mu_n} \lambda_j u(x_1, \ldots, x_{n-1}, -jx_n) & \text{if } x_n < 0. \end{cases}$$

If u is in addition ACL on $\mathbf{R}^n_+$, then Eu is defined to be ACL on $\mathbf{R}^n$ and

$$D^\mu(Eu) = E_\mu(D^\mu u)$$

whenever $|\mu| = 1$. Repeating this process, we find that

$$D^\mu(Eu) = E_\mu(D^\mu u) \qquad \text{whenever } |\mu| \leqq m.$$

Thus it follows that $Eu \in BL_m(L^p(\mathbf{R}^n))$.

Theorem 2.2. *If u is a function in $BL_m(L^p(\mathbf{R}^n_+))$, then there exists a polynomial P of degree at most $m-1$ such that*

$$u(x) = \sum_{|\mu|=m} a_\mu \int_{\mathbf{R}^n} k_{\mu,\ell}(x,y) E_\mu D^\mu u(y)dy + P(x) \qquad \text{a.e. on } \mathbf{R}^n_+.$$

3 Fine limits of BLD functions

Let G be an open set in $\mathbf{R}^n$. For a set E, we consider the relative capacity

$$C_{m,p}(E; G) = \inf \ \|f\|_p{}^p,$$

where the infimum is taken over all nonnegative measurable functions f on $\mathbf{R}^n$ such that $f = 0$ outside G and

$$U_m f(x) = \int |x-y|^{m-n} f(y)dy \geqq 1 \qquad \text{for every } x \in E.$$

It is easy to see that $C_{m,p}(\cdot; G)$ is a countably subadditive, nondecreasing outer capacity. Note further that in case $mp \geqq n$,

$$C_{m,p}(E; \mathbf{R}^n) = 0$$

for every set E; that is, $C_{m,p}(\mathbf{R}^n; \mathbf{R}^n) = 0$. Thus we write $C_{m,p}(E) = 0$ simply if

$$C_{m,p}(E \cap G; G) = 0 \qquad \text{for every open set } G.$$

It is not difficult to show that if $C_{m,p}(E; G) = 0$ for some bounded set G, then $C_{m,p}(E) = 0$.

Lemma 3.1. *If $C_{m,p}(E) = 0$, then there exists a nonnegative measurable function $f \in L^p(\mathbf{R}^n)$ such that $U_m f = \infty$ on E and*

$$\int_{\mathbf{R}^n} (1+|y|)^{m-n} f(y)dy < \infty. \tag{3.1}$$

Conversely, if

$$E_1 = \{x \in \mathbf{R}^n : U_m f(x) = \infty\}$$

for a nonnegative function $f \in L^p(\mathbf{R}^n)$ satisfying (3.1), *then $C_{m,p}(E_1) = 0$.*

Note here that $U_m f \not\equiv \infty$ on $\mathbf{R}^n$ if and only if (3.1) holds.

Lemma 3.2. *Let $mp = n$ and $f \in L^p(\mathbf{R}^n)$. If*

$$E_2 = \{\xi \in \partial\mathbf{R}^n_+ : \limsup_{r\to 0} \left(\log \frac{1}{r}\right)^{p-1} \int_{B(0,r)} |f(y)|^p dy > 0\},$$

then $C_{m,p}(E_2) = 0$, where $B(x,r)$ denotes the open ball centered at x with radius r (see Meyers [8], [9]).

Theorem 3.1. *Let $mp = n$ and f be a nonnegative function in $L^p(\mathbf{R}^n)$ satisfying* (3.1). *If $\xi \in \partial\mathbf{R}^n_+ - (E_1 \cup E_2)$, then there exists a set $E(\xi) \subseteqq \mathbf{R}^n_+$ such that*

$$\lim_{x\to\xi, x\in\mathbf{R}^n_+ - E(\xi)} U_m f(x) = U_m f(\xi) \tag{3.2}$$

and E is (m,p)-semithin at ξ, that is,

$$\lim_{r\to 0} \left(\log \frac{1}{r}\right)^{p-1} C_{m,p}(E(\xi) \cap B(\xi,r); B(\xi,2r)) = 0. \tag{3.3}$$

Proof. Write $U_m f(x) = u_1(x) + u_2(x)$, where

$$\begin{aligned} u_1(x) &= \int_{\mathbf{R}^n - B(x,|x-\xi|/2)} |x-y|^{m-n} f(y)dy, \\ u_2(x) &= \int_{B(x,|x-\xi|/2)} |x-y|^{m-n} f(y)dy. \end{aligned}$$

If $y \in \mathbf{R}^n - B(x,|x-\xi|/2)$, then

$$|\xi - y| \leqq |\xi - x| + |x-y| \leqq 3|x-y|,$$

so that Lebesgue's dominated convergence theorem implies that

$$\lim_{x\to\xi} u_1(x) = U_m f(\xi).$$

For each positive integer j, consider

$$E(j) = \{x : 2^{-j} \leqq |x-\xi| < 2^{-j+1}, u_2(x) > a_j^{-1/p}\},$$

where $\{a_j\}$ is a sequence of positive numbers such that $\lim_{j\to\infty} a_j = \infty$,

$$\lim_{j\to\infty} a_j j^{p-1} \int_{B(\xi,2^{-j+2})} |f(y)|^p dy = 0$$

and

$$\sum_{j=k}^{\infty} a_j \int_{G_j} |f(y)|^p dy \leqq 2a_k \sum_{j=k}^{\infty} \int_{G_j} |f(y)|^p dy,$$

where $G_j = \{x : 2^{-j-1} < |x-\xi| < 2^{-j+2}\}$. If $x \in E(j)$, then $B(x,|x-\xi|/2) \subseteqq G_j$ and thus

$$a_j^{-1/p} < \int_{G_j} |x-y|^{m-n} f(y)dy.$$

Hence it follows that

$$C_{m,p}(E(j);G_j) \leqq a_j \int_{G_j} |f(y)|^p dy.$$

Now define

$$E(\xi) = \bigcup_{j=1}^{\infty} E(j).$$

Then we have

$$\begin{aligned} C_{m,p}(E(\xi)\cap B(\xi,2^{-k+1}); B(\xi,2^{-k+2})) &\leqq \sum_{j=k}^{\infty} C_{m,p}(E(j);G_j) \\ &\leqq \sum_{j=k}^{\infty} a_j \int_{G_j} |f(y)|^p dy \\ &\leqq 6a_k \int_{B(\xi,2^{-k+2})} |f(y)|^p dy, \end{aligned}$$

which shows that

$$\lim_{k\to\infty} k^{p-1}C_{m,p}(E(\xi)\cap B(\xi,2^{-k+1}); B(\xi,2^{-k+2})) = 0.$$

We see readily that this is equivalent to (3.3), and hence $E(\xi)$ has all the required properties.

We say that u is (m,p)-quasicontinuous on an open set D if for any given $\varepsilon > 0$ and a bounded open set $G \subsetneqq D$, there exists an open set $\omega \subsetneqq G$ such that $C_{m,p}(\omega; G) < \varepsilon$ and u is continuous as a function on $G-\omega$. If $u \in BL_m(L^p_{loc}(D))$ is (m,p)-quasicontinuous on D, then u is said to be a BLD function on D. Note that for each $u \in BL_m(L^p_{loc}(D))$, there exists a BLD function on D which is equal to u almost everywhere on D.

In view of Theorem 2.2, $u \in BL_m(L^p(\mathbf{R}^n_+))$ is represented a.e. on $B(0,R)\cap\mathbf{R}^n_+$ as

$$u(x) = \sum_{|\mu|=m} a_\mu \int_{B(0,2R)} k_\mu(x-y)E_\mu D^\mu u(y)dy + v(x)$$

for some $v \in C^\infty(B(0,R))$. Hence Theorem 3.1 gives the following.

Corollary 3.1. *Let $mp = n$ and u be a BLD function in $BL_m(L^p(\mathbf{R}^n_+))$. Then there exists a set $E \subsetneqq \partial\mathbf{R}^n_+$ with the following properties :*

(i) $C_{m,p}(E) = 0$.

(ii) *For each $\xi \in \partial\mathbf{R}^n_+ - E$, there exists a set $E(\xi) \subsetneqq \mathbf{R}^n_+$ satisfying* (3.3) *for which* $\lim_{x\to\xi, x\in\mathbf{R}^n_+-E(\xi)} u(x)$ *exists and is finite.*

4 Monotone functions

We say that a continuous function u on $\mathbf{R}^n_+$ is monotone (in the sense of Lebesgue) if

$$\max_{\overline{G}} u = \max_{\partial G} u \qquad \text{and} \qquad \min_{\overline{G}} u = \min_{\partial G} u$$

hold for any relatively compact open set G in $\mathbf{R}^n_+$, where $\overline{G} = G\cup\partial G$ (see Vuorinen [22], [23]). If f is monotone on $(0,\infty)$ and $\xi \in \partial\mathbf{R}^n_+$, then it is clear that the function

$$u(x) = f(|x-\xi|)$$

is monotone on $\mathbf{R}^n_+$. Harmonic functions, (weak) solutions in a wider class of (non)linear elliptic partial differential equations and the coordinate functions of quasiregular mappings are monotone (see e.g. Gilbarg-Trudinger [5], Heinonen-Kilpeläinen-Martio [6], Reshetnyak [19], Serrin [20] and Vuorinen [23]).

The key of proving our theorem mentioned above is the following result.

THEOREM 4.1 (cf. [7, Remark, p.9] and [23, section 16]). *If u is monotone on $B(x,r)$ and $n-1<p\leqq n$, then*

$$|u(x)-u(y)|^p \leqq Mr^{p-n}\int_{B(x,r)} |\nabla u(z)|^p dz$$

whenever $y\in B(x,r/2)$, with a positive constant M independent of r.

PROOF. Assume that u is monotone on $B(x,r)$, and take $y\in B(x,r/2)$. Without loss of generality, we may assume that $u(x)<u(y)$. If $r/2<t<r$, then by monotonicity there exist $y(t)\in\partial B(x,t)$ and $x(t)\in\partial B(x,t)$ such that

$$u(x(t))\leqq u(x)<u(y)\leqq u(y(t)).$$

In view of Sobolev's imbedding theorem, we see that

$$|u(x(t))-u(y(t))|^p\leqq Mt^{p-(n-1)}\int_{\partial B(x,t)}|\nabla\ u(z)|^p dS(z).$$

By integration over the interval $(r/2,r)$, we obtain

$$|u(x)-u(y)|^p\int_{r/2}^{r} t^{-p+(n-1)}dt\leqq M\int_{r/2}^{r}\left(\int_{\partial B(x,t)}|\nabla\ u(z)|^p dS(z)\right)dt,$$

which proves the required inequality.

5 T_∞-limits

We begin with an estimate of $C_{1,n}$-capacity of balls.

LEMMA 5.1. *Let $\xi\in\partial\mathbf{R}^n_+$ and $x\in\mathbf{R}^n_+$. If $mp=n$, then*

$$C_{m,p}(B(x,x_n/2);B(\xi,2|x-\xi|))\sim[\log(2|x-\xi|/x_n)]^{1-p}.$$

PROOF. Let $f(y)=|x-y|^{-m}$ for $y\in B(\xi,2|x-\xi|)-B(x,x_n/2)$ and $f(y)=0$ elsewhere. If $z\in B(x,x_n/2)$ and $y\notin B(x,x_n/2)$, then $|z-y|\leqq|z-x|+|x-y|\leqq 2|x-y|$, so that

$$\int|z-y|^{m-n}f(y)dy\geqq 2^{m-n}\int_{B(\xi,2|x-\xi|)-B(x,x_n/2)}|x-y|^{-n}dy\geqq M\log(2|x-\xi|/x_n).$$

Hence it follows that

$$\begin{aligned}C_{m,p}(B(x,x_n/2);B(\xi,2|x-\xi|)) &\leqq \int_{B(\xi,2|x-\xi|)-B(x,x_n/2)}[f(y)/M\log(2|x-\xi|/x_n)]^p dy\\ &\leqq M[\log(2|x-\xi|/x_n)]^{1-p}.\end{aligned}$$

Conversely, take a nonnegative measurable function g such that $g = 0$ outside $B(\xi, 2|x-\xi|)$ and $U_m g \geqq 1$ on $B(x, x_n/2)$. Then

$$\begin{aligned}
\frac{1}{|B(x,x_n/2)|}\int_{B(x,x_n/2)} dz &\leqq \frac{1}{|B(x,x_n/2)|}\int_{B(x,x_n/2)}\left(\int_{B(\xi,2|x-\xi|)}|z-y|^{m-n}g(y)dy\right)dz \\
&\leqq \int_{B(\xi,2|x-\xi|)}\left(\frac{1}{|B(x,x_n/2)|}\int_{B(x,x_n/2)}|z-y|^{m-n}dz\right)g(y)dy \\
&\leqq M\int_{B(\xi,2|x-\xi|)}(x_n+|x-y|)^{m-n}g(y)dy \\
&\leqq M[\log(2|x-\xi|/x_n)]^{1/p'}\|g\|_p,
\end{aligned}$$

which proves that

$$C_{m,p}(B(x,x_n/2);B(\xi,2|x-\xi|)) \geqq M[\log(2|x-\xi|/x_n)]^{1-p}.$$

Theorem 5.1. *Let u be a monotone BLD function on $\mathbf{R}^n_+$ which belongs to $BL_1(L^n(\mathbf{R}^n_+))$. Then there exists a set $E \subseteqq \partial\mathbf{R}^n_+$ such that $C_{1,n}(E) = 0$ and u has a finite T_∞-limit at every boundary point $\xi \in \partial\mathbf{R}^n_+ - E$.*

Proof. For $\xi \in \partial\mathbf{R}^n_+ - (E_1 \cup E_2)$, take a set $E(\xi)$ as in Corollary 3.1. Since u is monotone on $\mathbf{R}^n_+$,

$$|u(x)-u(y)|^n \leqq M\int_{B(x,x_n)}|\text{grad } u(z)|^n dz \tag{5.1}$$

whenever $y \in B(x,x_n/2)$, where $x = (x_1,\ldots,x_n) \in \mathbf{R}^n_+$. If $x \in T_\gamma(\xi,a)$, then Lemma 5.1 implies that $B(x,x_n/2) - E(\xi)$ is not empty, so that there exists $y(x) \in B(x,x_n/2) - E(\xi)$ (when x_n is small enough). Then we see from (5.1) that

$$\lim_{x\to\xi,\,x\in T_\gamma(\xi,a)}|u(x)-u(y(x))| = 0.$$

Hence it follows that

$$\lim_{x\to\xi,\,x\in T_\gamma(\xi,a)} u(x) = \lim_{x\to\xi,\,x\in T_\gamma(\xi,a)} u(y(x)),$$

so that the limit on the left exists and is finite. Thus $E = E_1 \cup E_2$ has all the required properties, with the aid of Lemmas 3.1 and 3.2.

Corollary 5.1. *Every coordinate function of bounded quasiconformal mappings on $\mathbf{R}^n_+$ has a finite T_∞-limit at every boundary point except for a set $E \subset \partial\mathbf{R}^n_+$ such that $C_{1,n}(E) = 0$.*

6 Remarks

Remark 6.1. According to [15, Remark 5], for given $\gamma > 1$ and $a > 0$ there exists a harmonic function u on $\mathbf{R}^n_+$ with finite Dirichlet integral such that

(i) u has a nontangential limit at the origin.

(ii) $\limsup_{x \to 0, x \in T_{\gamma'}(0,a') - T_{\gamma'}(0,a)} u(x) = \infty$ for every $\gamma' > \gamma$ and $a' > a$.

This shows that the existence of nontangential limits may not always imply that of tangential limits.

Remark 6.2. By applying the same spirit as the construction of u in Remark 6.1, we give one more example of such u.

For $x^{(j)} = (2^{-j}, 0, ..., 0) \in \partial \mathbf{R}^n_+$ and $0 < r_j < 2^{-j-1}$, consider the sets

$$B_j = [B(x^{(j)}, 2^{-j-2} s_j) - B(x^{(j)}, r_j s_j)] - \mathbf{R}^n_+, \qquad \text{where } s_j = \left(\log \frac{1}{2^j r_j}\right)^{(2-n)/n}$$

Suppose $\{r_j\}$ is chosen so small that

$$\sum_j \left(\log \frac{1}{2^j r_j}\right)^{1-n} < \infty; \tag{6.1}$$

if this is the case, $B = \bigcup_j \mathbf{R}^n_+ \cap B(x^{(j)}, r_j)$ is called $C_{1,n}$-thin at the origin in the sense of [13]. Taking a sequence $\{a_j\}$ of positive numbers such that

$$\lim_{j \to \infty} a_j = \infty$$

and

$$\sum_j a_j{}^n \left(\log \frac{1}{2^j r_j}\right)^{1-n} < \infty, \tag{6.2}$$

we now define

$$f(y) = \begin{cases} a_j \left(\log \dfrac{1}{2^j r_j}\right)^{-1} |x^{(j)} - y|^{-1} & \text{when } y \in B_j, \\ 0 & \text{elsewhere,} \end{cases}$$

and

$$u(x) = \int_{\mathbf{R}^n} \frac{x_n - y_n}{|x - y|^n} f(y) dy, \qquad x = (x_1, ..., x_n), y = (y_1, ..., y_n).$$

Then, as in [13, Proposition], we can prove :

(i) u is a harmonic function on $\mathbf{R}_+^n$ with finite Dirichlet integral.

(ii) u has a nontangential limit at the origin.

(iii) $\lim_{j\to\infty} u(x^{(j)} + (0, ..., 0, r_j)) = \infty$.

To show (i) and (ii), we note by (6.2) that

$$\int f(y)^n dy \leqq M \sum_j a_j{}^n \left(\log \frac{1}{2^j r_j}\right)^{-n+1} < \infty$$

and

$$\begin{aligned} u(0) &= \int (-y_n)|y|^{-n} f(y) dy \\ &\leqq M \sum_j a_j \left(\log \frac{1}{2^j r_j}\right)^{-1} 2^{jn} \int_{B_j} (-y_n)|x^{(j)} - y|^{-1} dy \\ &\leqq M \sum_j a_j \left(\log \frac{1}{2^j r_j}\right)^{-n+1} < \infty. \end{aligned}$$

Finally we see that for $x \in \mathbf{R}_+^n \cap B(x^{(j)}, r_j)$,

$$u(x) \geqq M a_j \left(\log \frac{1}{2^j r_j}\right)^{-1} \int_{r_j s_j}^{2^{-j-2} s_j} (|x - x^{(j)}| + r)^{1-n} r^{-1} r^{n-1} dr \geqq M a_j,$$

which implies that

$$\lim_{x\to 0, x\in B} u(x) = \infty.$$

Remark 6.3. Let ω be a positive nonincreasing continuous function on the interval $(0, \infty)$ such that

$$\int_0^1 \omega(t)^{-1/(n-1)} t^{-1} dt < \infty. \tag{6.3}$$

If u is a monotone function on $\mathbf{R}_+^n$ satisfying

$$\int_{\mathbf{R}_+^n} |\nabla u(x)|^n \omega(|x|) dx < \infty, \tag{6.4}$$

then we can show that u has a finite T_∞-limit at the origin.

In this case, $0 \notin (E_1 \cup E_2)$ and thus apply Theorem 5.1. We also refer to [16] for harmonic functions.

Remark 6.4. Let ω be a positive nonincreasing continuous function on the interval $(0, \infty)$ for which (6.3) does not hold. Then there exists a monotone function u on $\mathbf{R}_+^n$ satisfying (6.4) such that u fails to have a finite T_∞-limit at the origin.

In fact, letting

$$f(r) = \int_r^2 \omega(t)^{-1/(n-1)} t^{-1} dt,$$

we may consider the function

$$u(x) = \log(f(|x|)/f(1))$$

for $|x| \leqq 1$; define $u(x) = 0$ otherwise. Then note that $u(0) = \infty$ and

$$|\nabla u(x)| = |f'(|x|)/f(|x|)|,$$

so that

$$\begin{aligned} \int |\nabla u(x)|^n \omega(|x|) dx &= M \int_0^1 |f'(r)/f(r)|^n \omega(r) r^{n-1} dr \\ &= M \int_0^1 f(r)^{-n} [-f'(r)] dr \\ &= M \int_{f(1)}^{\infty} t^{-n} dt < \infty. \end{aligned}$$

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